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Exact solutions for the plane problem in piezoelectric materials with an elliptic or a crack

Cun-Fa Gao*, Wei-Xun Fan

Department of Aircraft, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, P.R. China

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Abstract

Based on the complex potential approach, the two-dimensional problems in a piezoelectric material containing an elliptic hole subjected to uniform remote loads are studied. The explicit, closed-form solutions satisfying the exact electric boundary condition on the hole surface are given both inside and outside the hole. When the elliptic hole degenerates into a crack, the field intensity factors are obtained. It is shown that the stress intensity factors are the same as that of isotropic material, while the electric displacement intensity factor depends on both the material properties and the mechanical loads, but not on the electric loads. In other words, the uniform electric loads have no influence on the field singularities. It is also shown that the impermeable crack assumption used previously to simply the electric condition is not valid to crack problems in piezoelectric materials.

1. Introduction

For decades, piezoelectric materials have found wide application in the electronic and electromechanical industries. These materials, however, have a disadvantage such as brittleness. Thus the problems on fracture mechanics of piezoelectric materials have received much attention in the last few years. A wealth of theoretical results have been presented by Parton (1976), Deeg (1980), Pak (1990), Sosa and Pak (1990), Suo et al. (1992), Wang (1992), Sosa (1992, 1993), Pak and Tobin (1993), Park and Sun (1995), Beom and Atluri (1996), Gao and Barnett (1996), Yu and Qin (1996), Qin and Yu (1997), Zhong and Meguid (1997), and Zhao et al. (1997a, b). However, it should be noted that up to now, the exact solution satisfying the real electric boundary is still quite limited. This is because nearly all previous analyses mentioned above are based on an impermeable crack assumption, that is, the crack faces are assumed to be impermeable to electric field, hence the

* Corresponding author. E-mail: zzcae@dns.nuaa.edu.cn

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electric displacement vanishes inside the crack. Using this assumption, one will arrive at the following results: the stress intensity factors are the same as ones of isotropic materials while electric displacement intensity factor only depends on the electric load at infinity. Moreover, it is also found that the total energy release rate (Pak, 1990; Park and Sun, 1995) is always negative solely in the presence of electric loading. As pointed out by Park and Sun (1995), this contradicts available experiment observation. Thus, the validity of the impermeable crack assumption is discussed by Pak and Tobin (1993), Hao and Shen (1994), Dunn (1994), Park and Sun (1995), Zhang and Tong (1996), Kogan et al. (1996), and Sosa and Khutoryansky (1996).

In fact, the crack problems in a piezoelectric material should be considered as the electric inclusion problems since the dielectric constant of the air or vacuum inside the crack is not equal to zero. This implies that when the loads are given at infinity, the electric displacement on the crack faces is also given indirectly and is in general not equal to zero. Consequently, many investigators tried to obtain the solutions of the crack problem by examining the corresponding inclusions problem in a piezoelectric material. Recently Sosa and Khutoryansky (1996) used the series expansion method to address the plane problem of a transversely isotropic piezoelectric medium with an elliptic hole. Their results show that the electric displacement is constant when uniform mechanical and electric loads are applied at infinity. Kogan et al. (1996) studied the stress and electric displacement field of a spherical inclusion in a transversely isotropic piezoelectric material based on the complex potential. As a special case, they obtained the stress and electric displacement of a penny-shaped crack, and pointed out in the first time that the electric displacement is constant and the stresses are equal to zero everywhere solely in the presence of electric loading. However, it should be noted that since a lot of constant coefficients are piled up in the solutions given by Sosa and Khutoryansky (1996), Kogan et al. (1996), it is not easy to reveal some coupling relations between mechanical and electrical fields.

In the present paper we examine the plane problem of an elliptic hole or crack in a transversely isotropic piezoelectric solid subjected to uniform loads by using the Sosa and Khutoryansky's work (1996), and focus on developing a concise method. As a result, very simple solutions in the form are obtained. Through these solutions, one can clearly see the coupling between mechanical and electrical fields.

2. Basic formulation

Following Berlincourt et al. (1964), the general equations governing the thee-dimensional theory of piezoelectricity in the absence of body forces and free charges can be written as

$$\varepsilon_{ij} = s_{ijkl}\sigma_{kl} + g_{kij}D_k, \quad E_i = -g_{ikl}\sigma_{kl} + \beta_{ik}D_k \tag{1a,b}$$

$$\sigma_{ij,j} = 0 \quad D_{i,i} = 0 \tag{2a,b}$$

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{j,i} + u_{i,j}), \quad E_i = -\phi_i \tag{3a,b}$$

where $i, j, k, l = 1, 2, 3$; σ_{ij}, D_i, β_{ij}, u_i, E_i and ϕ are the components of stress, electric displacement, strain, displacement, electric field and electric potential, respectively; s_{ijkl}, g_{kij} and β_{ik} stand for elastic constants, piezoelectric constants, and dielectric constants, respectively; and a comma indicates partial derivative.

For a transversely isotropic solid referred to a Cartesian coordinate system x, y, z, assuming that x–y is the isotropic plane and z is the poling direction, then the constitutive eqn (1) can be simplified as

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ 2\varepsilon_{zy} \\ 2\varepsilon_{zx} \\ 2\varepsilon_{xy} \end{Bmatrix} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & 0 & 0 & 0 \\ s_{12} & s_{11} & s_{13} & 0 & 0 & 0 \\ s_{13} & {}_{13} & s_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & s_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(s_{11}-s_{12}) \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{zy} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix} + \begin{Bmatrix} 0 & 0 & g_{31} \\ 0 & 0 & g_{31} \\ 0 & 0 & g_{33} \\ 0 & g_{15} & 0 \\ g_{15} & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix} \begin{Bmatrix} D_x \\ D_y \\ D_z \end{Bmatrix} \tag{4}$$

$$\begin{Bmatrix} E_x \\ E_y \\ E_z \end{Bmatrix} = -\begin{bmatrix} 0 & 0 & 0 & 0 & g_{15} & 0 \\ 0 & 0 & 0 & g_{15} & 0 & 0 \\ g_{31} & g_{31} & g_{33} & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{zy} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix} + \begin{bmatrix} \beta_{11} & 0 & 0 \\ 0 & \beta_{11} & 0 \\ 0 & 0 & \beta_{33} \end{bmatrix} \begin{Bmatrix} D_x \\ D_y \\ D_z \end{Bmatrix} \tag{5}$$

eqns (2)–(5) constitute a set of basic equations for the three-dimensional problems of transversely isotropic piezoelectric media. In general, however, it is very difficult to analytically solve these equations due to the mathematical complexities. In order to explicitly study the electromechanical interaction, we only consider a special plane problem in this paper. If the plane of analysis is chosen to be the x–y plane, it is clear from eqns (4) and (5) that the inplane electric fields couple only with the out-of-plane elastic fields. This is a so-called antiplane strain problem (Pak, 1990; Zhang and Tong, 1996; Zhong and Meguid, 1997). A more complete state of electromechanical interaction can be observed if the plane of analysis is chosen to be the x–z plane or the y–z plane. In the present work we choose the former, i.e., the plane strain problem in the x–z plane is considered for the study of plane electromechanical phenomena. Moreover, for the purpose of comparison with previous works (Sosa, 1991, 1992; Sosa and Khutoryansky, 1996), we further assume that

$$\varepsilon_{yy} = \varepsilon_{zy} = \varepsilon_{xy} = E_y = 0$$

and rename the coordinates: $x \to x_1$, $z \to x_2$. In this case eqns (4) and (5) can finally be simplified as (Sosa, 1991)

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{12} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{bmatrix} 0 & b_{21} \\ 0 & b_{22} \\ b_{13} & 0 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} \tag{6}$$

$$\begin{Bmatrix} E_1 \\ E_2 \end{Bmatrix} = -\begin{bmatrix} 0 & 0 & b_{13} \\ b_{21} & b_{22} & 0 \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} + \begin{bmatrix} \delta_{11} & 0 \\ 0 & \delta_{22} \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} \tag{7}$$

where

$$a_{11} = s_{11} - \frac{s_{12}^2}{s_{11}}, \quad a_{12} = s_{13} - \frac{s_{12}s_{13}}{s_{11}}, \quad a_{22} = s_{33} - \frac{s_{13}^2}{s_{11}}, \quad a_{33} = s_{44}$$

$$b_{21} = \left(1 - \frac{s_{12}}{s_{11}}\right) g_{31}, \quad b_{22} = g_{33} - \frac{s_{13}}{s_{11}} g_{31}, \quad b_{13} = g_{15}, \quad \delta_{11} = \beta_{11}, \quad \delta_{22} = \beta_{33} + \frac{g_{31}^2}{s_{11}}$$

Now we have reduced a three-dimensional problem to a two-dimensional one governed by eqns (6), (7), (2) and (3). Following Sosa (1991), the field solutions for the two-dimensional problem can be expressed, respectively, as

$$\langle \sigma_{22}, \sigma_{12}, \sigma_{11} \rangle = 2\,\mathrm{Re} \sum_{k=1}^{3} \langle 1, -\mu_k, \mu_k^2 \rangle \phi_k(z_k) \quad z_k = x_1 + \mu_k x_2 \tag{8}$$

$$\langle u_1, u_2 \rangle = 2\,\mathrm{Re} \sum_{k=1}^{3} \langle p_k, q_k \rangle \varphi_k(z_k), \quad \mathrm{Im}\,\mu_k > 0 \tag{9}$$

$$\langle E_1, E_2 \rangle = 2\,\mathrm{Re} \sum_{k=1}^{3} \kappa_k \langle 1, \mu_k \rangle \phi_k(z_k) \tag{10}$$

$$\langle D_1, D_2 \rangle = 2\,\mathrm{Re} \sum_{k=1}^{3} \lambda_k \langle \mu_k, -1 \rangle \varphi_k(z_k) \tag{11}$$

$$\phi(z_k) = -2\,\mathrm{Re} \sum_{k=1}^{3} \kappa_k \varphi_k(z_k)$$

$$p_k = a_{11}\mu_k^2 + a_{12} - b_{21}\lambda_k, \quad q_k = (a_{12}\mu_k^2 + a_{22} - b_{22}\lambda_k)/\mu_k$$

$$\lambda_k = -\frac{(b_{21} + b_{31})\mu_k^2 + b_{22}}{\delta_{11}\mu_k^2 + \delta_{22}}, \quad \kappa_k = (b_{13} + \delta_{11}\lambda_k)\mu_k, \phi_k(z_k) = \mathrm{d}\varphi(z_k)/z_k \tag{12}$$

where Re and Im denote the real and imaginary part; μ_k are distinct complex parameters to be determined by the characteristic equation (Sosa, 1991), and $\varphi_k(z_k)$ are the three complex potentials to be determined.

To find $\varphi_k(z_k)$, one can use the following boundary conditions:

$$2\,\mathrm{Re} \sum_{k=1}^{3} \varphi_k(z_k) = -\int_0^s t_{2s}\,\mathrm{d}s \tag{13}$$

$$2\,\mathrm{Re} \sum_{k=1}^{3} \mu_k \varphi_k(z_k) = \int_0^s t_{1s}\,\mathrm{d}s \tag{14}$$

$$2\,\mathrm{Re} \sum_{k=1}^{3} \lambda_k \varphi_k(z_k) = \int_0^s D_n\,\mathrm{d}s \tag{15}$$

in which t_{1s} and t_{2s} represent the x_1 and x_2 components of the force, respectively; s is the arc-length on the boundary; D_n is the normal component of electric displacement.

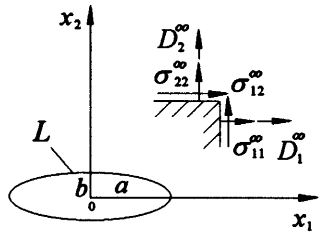

Fig. 1. An elliptic hole in a piezoelectric solid subjected to uniform loads at infinity.

For later use, we introduce the following matrix notation:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ \mu_1 & \mu_2 & \mu_3 \\ \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix}$$

It can be proved that A is nonsingular when μ_k are distinct. Below, we denote the inverse of A by

$$\Lambda = (\Lambda_{kj})_{3\times 3} = A^{-1}$$

then we have the invertibility relation:

$$\sum_{k=1}^{3} A_{ik}\Lambda_{kj} = \delta_{ij}^0 \tag{16}$$

where $\delta_{ij}^0 = 0$ for $i \neq j$ or $\delta_{ij}^0 = 1$ for $i = j$.

3. The solution to an elliptic hole

As in Sosa (Sosa, 1991; Sosa and Khutoryansky, 1996), we consider the two-dimensional problem in a transversely isotropic solid containing an elliptic hole, with semi-axes a and b directed along x_1- and x_2-axes, respectively (see Fig. 1). The uniform stress $\sigma_{11}^\infty, \sigma_{12}^\infty, \sigma_{22}^\infty$ and electric displacement D_1^∞, D_2^∞ are given at infinity. In addition, the hole is assumed to be free of forces and external charges, but to be filled with homogeneous air.

3.1. Field solutions inside the hole L

Following Sosa and Khutoryansky's study (1996), we assume that the electric potential, $\phi_0(x_1, x_2)$, inside L is of the following form:

$$\phi_0(x_1, x_2) = -e_1x_1 - e_2x_2 \tag{17}$$

where e_1 and e_2 are two real constants.

From eqn (17), the electric field components (E_1^0, E_2^0) and the electric displacement components (D_1^0, D_2^0) inside L can be expressed as:

$$E_1^0 = -\frac{\partial \phi_0}{\partial x_1} = e_1, \quad E_2^0 = -\frac{\partial \phi_0}{\partial x_2} = e_2,$$

$$D_1^0 = \varepsilon_0 E_1^0, \quad D_2^0 = \varepsilon_0 E_2^0 \tag{18a–d}$$

where ε_0 is the dielectric constant air inside the hole.

3.2. Field solutions in the material

For this problem, the complex potential in the material can be written as

$$\varphi_k(z_k) = B_k z_k + \varphi_{k_0}(z_k) \tag{19}$$

where $\varphi_{k_0}(z_k)$ is a holomorphic function outside L_k in the z_k-plane (L_k is obtained from L in z-plane by the affine transformation: $z_k = x_1 + \mu_k x_2$) up to infinity ($z_k = \infty$) and $\varphi_{k_0}(\infty) = 0$; B_k are complex constants to be determined from the far field loading conditions (Sosa, 1991).

Obviously the first term on the right of eqn (19) stands for the complex potential of an infinite piezoelectric medium without hole subjected to the uniform loads at infinity. Using eqns (13)–(15) and noting that in this case, the electro-elastic fields in the piezoelectric medium are the same everywhere as those applied at infinity, one has on any arc-length that

$$2\,\mathrm{Re}\sum_{k=1}^{3} B_k z_k = -\int_0^s t_{2s}^{\infty}\,\mathrm{d}s \tag{20a}$$

$$2\,\mathrm{Re}\sum_{k=1}^{3} \mu_k B_k z_k = \int_0^s t_{1s}^{\infty}\,\mathrm{d}s \tag{20b}$$

$$2\,\mathrm{Re}\sum_{k=1}^{3} \lambda_k B_k z_k = \int_0^s D_n^{\infty}\,\mathrm{d}s \tag{20c}$$

where t_{1s}^{∞}, t_{2s}^{∞} and D_n^{∞} represent the components of the force and electric displacement in a piezoelectric medium without hole when the medium is subjected to the remote uniform loads, and it is easy to prove that these components are related to the remote loads through the following equations

$$-\int_0^s t_{2s}^{\infty}\,\mathrm{d}s = -\int_0^s \sigma_{12}^{\infty}\,\mathrm{d}x_2 - \sigma_{22}^{\infty}\,\mathrm{d}x_1$$

$$\int_0^s t_{1s}^{\infty}\,\mathrm{d}s = \int_0^s \sigma_{11}^{\infty}\,\mathrm{d}x_2 - \sigma_{12}^{\infty}\,\mathrm{d}x_1$$

$$\int_0^s D_n^{\infty}\,\mathrm{d}s = \int_0^s D_1^{\infty}\,\mathrm{d}x_2 - D_2^{\infty}\,\mathrm{d}x_1$$

On the hole surface, the boundary conditions are

$$\int_L t_{2s}\,\mathrm{d}s = 0$$

$$\int_L t_{1s}\,\mathrm{d}s = 0$$

$$\int_L D_n\,\mathrm{d}s = \int_L D_n^0\,\mathrm{d}s = \int_L D_1^0\,\mathrm{d}x_2 - D_2^0\,\mathrm{d}x_1$$

$$\phi = \phi_0 \tag{21a–d}$$

and there are the following relations:

$$x_1 = a\,\mathrm{con}\,\theta, x_2 = b\sin\theta, con\,\theta = \frac{1}{2}\left(\sigma + \frac{1}{\sigma}\right), \quad \sin\theta = \frac{i}{2}\left(\frac{1}{\sigma} - \sigma\right), \quad \mathrm{d}\theta = \frac{\mathrm{d}\sigma}{i\sigma} \tag{22a–e}$$

where $\sigma = \exp(i\theta)$, θ is the angle measured over a unit circle in counter-clockwise direction.

Substituting eqn (22) into eqn (21c) yields

$$\int_L D_n^0\,\mathrm{d}s = -\frac{1}{2}\left[aD_2^0\left(\sigma + \frac{1}{\sigma}\right) + ibD_1^0\left(\sigma - \frac{1}{\sigma}\right)\right] \tag{23}$$

Substituting eqns (19) and (21a–c) into eqns (13)–(15), and using eqns (20) leads to

$$-\int_L t_{2s}^{\infty}\,\mathrm{d}s + 2\,\mathrm{Re}\sum_{k=1}^{3}\varphi_{k_0}(z_k) = 0$$

$$\int_L t_{1s}^{\infty}\,\mathrm{d}s + 2\,\mathrm{Re}\sum_{k=1}^{3}\mu_k\varphi_{k_0}(z_k) = 0$$

$$\int_L D_n^{\infty}\,\mathrm{d}s + 2\,\mathrm{Re}\sum_{k=1}^{3}\lambda_k\varphi_{k_0}(z_k) = \int_L D_n^0\,\mathrm{d}s, \quad \text{on } L \tag{24a–c}$$

where

$$-\int_L t_{2s}^{\infty}\,\mathrm{d}s = -\int_L \sigma_{12}^{\infty}\,\mathrm{d}x_2 - \sigma_{22}^{\infty}\,\mathrm{d}x_1$$

$$\int_L t_{1s}^{\infty}\,\mathrm{d}s = \int_L \sigma_{11}^{\infty}\,\mathrm{d}x_2 - \sigma_{12}^{\infty}\,\mathrm{d}x_1$$

$$\int_L D_n^{\infty}\,\mathrm{d}s = \int_L D_1^{\infty}\,\mathrm{d}x_2 - D_2^{\infty}\,\mathrm{d}x_1 \tag{25a–c}$$

Substituting eqn (22) into eqn (25) results in

$$-\int_L t_{2s}^{\infty}\,\mathrm{d}s = \frac{1}{2}\left[a\sigma_{22}^{\infty}\left(\sigma + \frac{1}{\sigma}\right) + ib\sigma_{12}^{\infty}\left(\sigma - \frac{1}{\sigma}\right)\right]$$

$$\int_L t_{1s}^{\infty}\,\mathrm{d}s = -\frac{1}{2}\left[a\sigma_{12}^{\infty}\left(\sigma+\frac{1}{\sigma}\right)+ib\sigma_{11}^{\infty}\left(\sigma-\frac{1}{\sigma}\right)\right]$$

$$\int_L D_n^{\infty}\,\mathrm{d}s = -\frac{1}{2}\left[aD_2^{\infty}\left(\sigma+\frac{1}{\sigma}\right)+ibD_1^{\infty}\left(\sigma-\frac{1}{\sigma}\right)\right] \tag{26a–c}$$

Let us now introduce the following mapping function $z_k(\zeta_k)$:

$$z_k(\zeta_k) = R_k(\zeta_k+m_k\zeta_k^{-1}),\quad R_k = \frac{a-\mu_k b}{2},\quad m_k = \frac{a+\mu_k b}{a-\mu_k b} \tag{27a–c}$$

which transforms the exterior of the elliptic L_k in the z_k-plane into the exterior of a unit circle γ, which is described by $\zeta_k = \sigma$, in the complex plane ζ_k, and then substitute eqns (23), (26) and (27) into eqn (24), we have

$$2\,\mathrm{Re}\sum_{k=1}^{3}\Phi_{k_0}(\sigma) = -\frac{1}{2}\left[a\sigma_{22}^{\infty}\left(\sigma+\frac{1}{\sigma}\right)+ib\sigma_{12}^{\infty}\left(\sigma-\frac{1}{\sigma}\right)\right]$$

$$2\,\mathrm{Re}\sum_{k=1}^{3}\mu_k\Phi_{k_0}(\sigma) = \frac{1}{2}\left[a\sigma_{12}^{\infty}\left(\sigma+\frac{1}{\sigma}\right)+ib\sigma_{11}^{\infty}\left(\sigma-\frac{1}{\sigma}\right)\right]$$

$$2\,\mathrm{Re}\sum_{k=1}^{3}\lambda_k\Phi_{k_0}(\sigma) = \frac{1}{2}\left[a(D_2^{\infty}-D_2^{0})\left(\sigma+\frac{1}{\sigma}\right)+ib(D_1^{\infty}-D_1^{0})\left(\sigma-\frac{1}{\sigma}\right)\right] \tag{28a–c}$$

where a notation $\Phi_{k_0}(\zeta_k) = \varphi_{k_0}[z_k(\zeta_k)]$ is introduced.

Multiplying both sides of eqn (28) by $\mathrm{d}\sigma/(\sigma-\zeta_k)$ and integrating over the whole unit circle (Muskhelishvili, 1975), we obtain

$$\sum_{k=1}^{3}\Phi_{k_0}(\zeta_k) = -\frac{1}{2}[a\sigma_{22}^{\infty}-ib\sigma_{12}^{\infty}]\zeta_k^{-1}$$

$$\sum_{k=1}^{3}\mu_k\Phi_{k_0}(\zeta_k) = \frac{1}{2}[a\sigma_{12}^{\infty}-ib\sigma_{11}^{\infty}]\zeta_k^{-1}$$

$$\sum_{k=1}^{3}\lambda_k\Phi_{k_0}(\zeta_k) = \frac{1}{2}[a(D_2^{\infty}-D_2^{0})-ib(D_1^{\infty}-D_1^{0})]\zeta_k^{-1} \tag{29a–c}$$

Solving eqn (29) yields

$$\varphi_{k_0}(z_k) = a_{k_1}\zeta_k^{-1}(z_k) \tag{30}$$

where

$$a_{k_1} = \sum_{j=1}^{3}\Lambda_{kj}F_j$$

$$F = \tfrac{1}{2}[-(a\sigma_{22}^{\infty}-ib\sigma_{12}^{\infty}),(a\sigma_{12}^{\infty}-ib\sigma_{11}^{\infty}),a(D_2^{\infty}-D_2^{0})-ib(D_1^{\infty}-D_1^{0})]^T \tag{31a,b}$$

On the other hand, substituting eqns (12) together with eqns (19), (30), and (17) together with (22) into (21d), we have, with a little arrangement, that

$$\sigma^2\overline{c_k}+c_k = 0 \tag{32}$$

where

$$c_k = 2\sum_{k=1}^{3}(\kappa_k B_k R_k m_k+\overline{\kappa_k B_k R_k}+\kappa_k a_{k_1})-(ae_1-ibe_2) \tag{33}$$

Inserting eqns (31) and (27b,c) into eqn (33), and using eqn (10) leads to

$$c_k = aE_1^\infty+ibE_2^\infty+[a(D_2^\infty-D_2^0)-ib(D_1^\infty-D_1^0)]\sum_{j=1}^{3}\kappa_j\Lambda_{j_3}+2\sum_{k=1}^{3}\sum_{j=1}^{2}\kappa_k\Lambda_{kj}F_j-(ae_1-ibe_2) \tag{34}$$

Since all the points on the unit are the root of eqn (32), this implies that $c_k = 0$, namely

$$\operatorname{Re} c_k = 0, \quad \operatorname{Im} c_k = 0 \tag{35a,b}$$

Substituting eqn (34) into eqn (35) yields

$$aE_1^\infty+a(D_2^\infty-D_2^0)C_\mathrm{R}+b(D_1^\infty-D_1^0)C_\mathrm{I}+2\operatorname{Re}\sum_{k=1}^{3}\sum_{j=1}^{2}\kappa_k\Lambda_{kj}F_j-aE_1^0 = 0$$

$$bE_2^\infty+a(D_2^\infty-D_2^0)C_\mathrm{I}-b(D_1^\infty-D_1^0)C_\mathrm{R}+2\operatorname{Im}\sum_{k=1}^{3}\sum_{j=1}^{2}\kappa_k\Lambda_{kj}F_j+bE_2^0 = 0 \tag{36a,b}$$

where

$$c_\mathrm{R} = \operatorname{Re}\sum_{j=1}^{3}\kappa_j\Lambda_{j_3}, \quad c_\mathrm{I} = \operatorname{Im}\sum_{j=1}^{3}\kappa_j\Lambda_{j_3} \tag{37a,b}$$

Using eqn (18c,d), eqn (36) can be reduced to

$$(D_2^\infty-D_2^0)a\varepsilon_0 C_\mathrm{R}+(D_1^\infty-D_1^0)(a+b\varepsilon_0)C_\mathrm{I} = b_1$$

$$(D_2^\infty-D_2^0)(b-a\varepsilon_0)c_\mathrm{I}+(D_1^\infty-D_1^0)b\varepsilon_0 c_\mathrm{R} = b_2 \tag{38a,b}$$

where

$$b_1 = aD_1^\infty-a\varepsilon_0 E_1^\infty-2\varepsilon_0\operatorname{Re}\sum_{k=1}^{3}\sum_{j=1}^{2}\kappa_k\Lambda_{kj}F_j$$

$$b_2 = bD_2^\infty+b\varepsilon_0 E_2^\infty+2\varepsilon_0\operatorname{Im}\sum_{k=1}^{3}\sum_{j=1}^{2}\kappa_k\Lambda_{kj}F_j$$

Solving eqn (38), one obtains

$$D_2^\infty-D_2^0 = \frac{b\varepsilon_0 c_\mathrm{R}b_1-(a+b\varepsilon_0)c_\mathrm{I}b_2}{\Delta}$$

$$D_1^\infty - D_1^0 = \frac{a\varepsilon_0 c_R b_2 - (b - a\varepsilon_0)c_I b_1}{\Delta} \tag{39a,b}$$

with

$$\Delta = ab\varepsilon_0 c_R^2 - (a + b\varepsilon_0)(b - a\varepsilon_0)c_I^2 \tag{40}$$

Substituting eqns (30) and (31a) into (19), and using (27) leads to

$$\varphi_k(z_k) = B_k z_k + [\Lambda_{k_1}F_1 + \Lambda_{k_2}F_2 + \Lambda_{k_3}F_3]\frac{z_k - \sqrt{z_k^2 - (a^2 + \mu_k^2 b^2)}}{a + i\mu_k b} \tag{41}$$

in which F_j is determined by eqns (31b) and (39).

Taking derivative in eqn (41) with respect to z_k results in

$$\phi_k(z_k) = B_k + [\Lambda_{k_1}F_1 + \Lambda_{k_2}F_2 + \Lambda_{k_3}F_3]\frac{1}{a + i\mu_k b}\left[1 - \frac{z_k}{\sqrt{z_k^2 - (a^2 + \mu_k^2 b^2)}}\right] \tag{42}$$

4. The solution to a crack

4.1. The field solutions inside the crack

When the elliptic hole degenerate into a crack, let $b = 0$, then eqns (40) and (39a) become

$$\Delta = a^2 \varepsilon_0 c_I^2$$

$$D_2^\infty - D_2^0 = \frac{-2a\varepsilon_0 c_I \operatorname{Im} \sum_{k=1}^{3} \sum_{j=1}^{2} \kappa_k \Lambda_{kj} F_j}{\Delta} \tag{43a,b}$$

Inserting eqns (31b), (37b) and (43a) into eqn (43b) yields

$$D_2^\infty - D_2^0 = \frac{\sigma_{22}^\infty \operatorname{Im} \sum_{k=1}^{3} \kappa_k \Lambda_{k_1} - \sigma_{12}^\infty \operatorname{Im} \sum_{k=1}^{3} \kappa_k \Lambda_{k_2}}{\operatorname{Im} \sum_{k=1}^{3} \kappa_k \Lambda_{k_3}} \tag{44}$$

Noting that $\Sigma_{k=1}^{3} \kappa_k \Lambda_{k_2}$ is real (see Appendix), eqn (44) becomes

$$D_2^\infty - D_2^0 = \frac{\operatorname{Im} \sum_{k=1}^{3} \kappa_k \Lambda_{k_1}}{\operatorname{Im} \sum_{k=1}^{3} \kappa_k \Lambda_{k_3}} \sigma_{22}^\infty \tag{45}$$

Equation (45) shows that the normal component of electric displacement on the crack faces is not equal to zero, which depends on both mechanical loads and electric loads.

On the other hand, letting $b = 0$ in eqn (36a) leads to

$$E_1^0 = E_1^\infty - \mathrm{Re}\sum_{k=1}^{3}\kappa_k[\Lambda_{k_1}\sigma_{22}^\infty - \Lambda_{k_2}\sigma_{12}^\infty - \Lambda_{k_3}(D_2^\infty - D_2^0)] \tag{46}$$

in which $D_2^\infty - D_2^0$ is determined by eqn (45).

Using eqn (18), one can obtain the other two components

$$E_2^0 = \frac{D_2^0}{\varepsilon_0}, \quad D_1^0 = \varepsilon_0 E_1^0 \tag{47}$$

4.2. The field solutions in the piezoelectric material

Letting $b = 0$ in eqns (42) and (31b) gives

$$\phi_k(z_k) = B_k - \frac{f_{k_0} z_k}{2\sqrt{z_k^2 - a^2}} + \frac{1}{2} f_{k_0} \tag{48}$$

where

$$f_{k_0} = -[\Lambda_{k_1}\sigma_{22}^\infty - \Lambda_{k2}\sigma_{12}^\infty - \Lambda_{k_3}(D_2^\infty - D_2^0)] \tag{49}$$

Equation (48) together with eqns (49) and (45) shows that $\phi_k(z_k)$ is independent of ε_0. This means the field solutions in the material are not related to ε_0.

According to the convectional definition, the field singularity coefficients at the crack tip, $x_1 = a$, can be expressed as

$$(k_1, k_2, k_\mathrm{D}) = \sqrt{2\pi}\lim_{x_1 \to a}(x_1 - a)^{1/2}(\sigma_{22}, \sigma_{12}, D_2) \tag{50}$$

Substituting eqns (8) and (11) into (50), one obtains

$$(k_1, k_2, k_\mathrm{D}) = 2\sqrt{2\pi}\,\mathrm{Re}\lim_{x_1 \to a}(x_1 - a)^{1/2}\sum_{k=1}^{3}\phi_k(x_1)(1, -\mu_k, -\lambda_k) \tag{51}$$

Substituting eqns (48) and (49) into eqn (51), and using eqn (16) gives

$$(k_1, k_2, k_\mathrm{D}) = \sqrt{\pi a}(\sigma_{22}^\infty, \sigma_{12}^\infty, D_2^\infty - D_2^0) \tag{52}$$

If assuming that $D_2^0 = 0$ in eqn (52), one has

$$k_\mathrm{D} = \sqrt{\pi a} D_2^\infty$$

which is the result obtained previously according to the impermeable crack assumption. Nevertheless, inserting eqn (45) into (52) yields

$$k_\mathrm{D} = \frac{\mathrm{Im}\sum_{k=1}^{3}\kappa_k\Lambda_{k_1}}{\mathrm{Im}\sum_{k=1}^{3}\kappa_k\Lambda_{k_3}} k_1 \tag{53}$$

Equation (52) and (53) show that the stress intensity factors in a piezoelectric material are the same as that of a common material while the electric displacement intensity factor depends on material properties and the mechanical loads, but not on the electric loads. Especially when there are only electric loads at infinity, one can see from eqns (48), (49) and (45) that

$$\phi_k(z_k) = B_k \tag{54}$$

Equation (54) implies that in this case, the stress field vanishes and the electric field is uniform in the material. Inside the crack, the electric field components are

$$E_1^0 = E_1^\infty, \quad E_2^0 = \frac{D_2^0}{\varepsilon_0}$$

5. Conclusions

The complex potential method is used to analyze the plane problem in piezoelectric materials with an elliptic hole or crack. Exact solutions of the field intensity factors are presented. The solutions are very concise in form and satisfy the real electric boundary condition. It is shown that the stress intensity factors are the same as those obtained based on the impermeable crack assumption, while the electric displacement intensity factor is different from the previous results, it depends on both material properties and the mechanical loads, but not the electric loads. Especially when the mechanical loads vanish at infinity, the solution in this case is given by the uniform electric field and zero stresses everywhere in the material. In summary, the following conclusions may be drawn:

(1) The normal component of electric displacement on the crack faces is not equal to zero, which depends on the electric loads, mechanical loads and material properties.
(2) The electric loads have no influence on the field singularities.
(3) The singularity of electric displacement at crack tips depends on that of the stress.
(4) The field solution in piezoelectric materials are not related to the dielectric constant, ε_0, of air inside a crack, i.e. ε_0 has influence only on the electric field inside the crack.
(5) The impermeable crack assumption is not valid in solving the fracture problems in piezoelectric materials.

Appendix

$$\begin{aligned}\sum_{k=1}^{3} \kappa_k \Lambda_{k_2} &= \sum_{k=1}^{3} (b_{13}+\delta_{11}\lambda_k)\mu_k\Lambda_{k_2} \\ &= b_{13}\sum_{k=1}^{3} \mu_k\Lambda_{k_2}+\delta_{11}\sum_{k=1}^{3} \lambda_k\mu_k\Lambda_{k_2}\end{aligned} \tag{A1}$$

where

$$\Lambda = [\Lambda_{kj}] = \frac{1}{\Delta}\begin{bmatrix} \mu_2\lambda_3-\mu_3\lambda_2 & \lambda_2-\lambda_3 & \mu_3-\mu_2 \\ \mu_3\lambda_1-\mu_1\lambda_3 & \lambda_3-\lambda_1 & \mu_1-\mu_3 \\ \mu_1\lambda_2-\mu_2\lambda_1 & \lambda_1-\lambda_2 & \mu_2-\mu_1 \end{bmatrix} \tag{A2}$$

$$\Delta = (\lambda_2-\lambda_3)\mu_1+(\lambda_3-\lambda_1)\mu_2+(\lambda_1-\lambda_2)\mu_3 \tag{A3}$$

It is easy to confirm from eqn (A2) that:

$$\sum_{k=1}^{3} \mu_k \Lambda_{k_2} = 1 \tag{A4}$$

$$\sum_{k=1}^{3} \lambda_k\mu_k\Lambda_{k_2} = \frac{1}{\Delta}[(\mu_1\lambda_1\lambda_2-\mu_1\lambda_1\lambda_3)+(\mu_2\lambda_2\lambda_3-\mu_3\lambda_2\lambda_3)-(\mu_2\lambda_2\lambda_1-\mu_3\lambda_3\lambda_1)] \tag{A5}$$

For transversely isotropic piezoelectric media, Sosa and Khutoryansky's study (1996) shows that

$$\mathrm{Re}\,\mu_1 = \mathrm{Im}\,\lambda_1 = 0, \mu_3 = -\overline{\mu_2}, \lambda_3 = \overline{\lambda_2} \tag{A6}$$

Using eqn (A6), eqns (A5) and (A3) can be reduced to

$$\sum_{k=1}^{3} \lambda_k\mu_k\Lambda_{k_2} = \frac{1}{\Delta}[(\mu_1\lambda_2\lambda_2+\overline{\mu_1\lambda_1\lambda_2})+(\mu_2\lambda_2\overline{\lambda_2}+\overline{\mu_2\lambda_2}\lambda_2)-(\mu_2\lambda_2\lambda_1+\overline{\mu_2\lambda_2\lambda_1})] \tag{A7}$$

$$\Delta = (\lambda_2\mu_1+\overline{\lambda_2\mu_1})+(\overline{\lambda_2}\mu_2+\lambda_2\overline{\mu_2})-(\lambda_1\mu_2+\overline{\lambda_1\mu_2}) \tag{A8}$$

From eqns (A1), (A4), (A7) and (A8), one finds that $\Sigma_{k=1}^{3}\kappa_k\Lambda_{k_2}$ is real.

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